A technique for loop calculations in non-Abelian gauge theories –with application to five gluon amplitude–

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Abstract

A powerful tool for calculations in non-Abelian gauge theories is obtained by combining the background field gauge, the helicity basis and the color decomposition methods. It has reproduced the one-loop calculation of the five-gluon amplitudes in QCD, is applicable to electroweak processes and extendable to two-loop calculations.

1 Introduction

In search of new physics beyond the standard model that might be hiding in jet events at high energy collider experiments, it is important to understand the back ground arising from the known physics of the standard model. Here the conventional perturbation methods, which were so useful in establishing the standard model itself, face a new challenge. Calculations for the multi-jet processes often induce vast numbers of Feynman diagrams. And to make the matter worse, the complicated structure of the non-Abelian gauge-theory vertices amplify the number of terms in the intermediate stages of calculations so as to make them prohibitively difficult even at the tree level. In this paper we present a combination of the background field gauge, color decomposition and spinor helicity basis which makes loop calculations of such multi-jet processes feasible.

Parke et.al. developed methods to simplify multi-gluon amplitudes in the tree level [1]. In their methods, gluon amplitudes are decomposed into color-ordered subamplitudes [2] which are described by the spinor helicity basis [3]. These techniques came from the analogy to the string theory. The subamplitudes constructed by above methods are gauge invariant and they possess the important relationships which are well known as the dual Ward identity. Supersymmetry is also useful in the pQCD calculations[4].

The string-motivated technique was pushed one step further by Bern and Kosower and their coworkers [5, 6, 7]. They introduced a new technique, which is called the string-inspired method or Bern-Kosower Rule, to compute the one-loop pQCD amplitudes. This technique is based on the technology of string theory. The idea is that string amplitudes include the pQCD amplitudes in the infinite string tension limit. Using this technique, they performed the one loop calculation for the process $gg \to gg$ as a non trivial example[5, 6]. Their results agree with the results of the conventional calculations obtained by Ellis and Sexton[8]. They also gave the first calculation of the one-loop amplitude for five external gluons[7]. It is one of the most difficult part in the calculation of the next to leading order contribution for the three jets production processes[9]. No one has done it in the conventional Feynman diagram formula so far.

It is well known that, at the tree level, the string inspired method is connected with the Gervais-Neveu non-linear gauge[12]. However, the relation between the one-loop level string motivated calculation and the conventional pQCD calculation is still subtle. Many people suggested that there is a relation between the string inspired method and the background field gauge[10]. The background field method and/or gauge is another powerful method. In the early works by DeWitt, the background field method was formulated to compute the quantum corrections for the effective action without losing explicit gauge invariance[13, 14]. In this gauge, we can construct the gauge invariant effective action $\Gamma[B]$, which is invariant under the gauge transformation of the classical background field B_μ. It was developed by Abbot in the QCD case[15]. He applied this method to compute the 1PI Green's functions. He also explained that the correct S-matrix is given from trees of 1PI Green's functions constructed in the background field gauge[16]. This means that the background field gauge allows us to calculate the S-matrix in a gauge invariant way.

The background field gauge is a conventional method based on the Feynman diagram formula. This method possesses several advantages to carry out loop calculations of the QCD amplitudes. In the following we demonstrate that this background field gauge combined with color decomposition and spinor helicity techniques pro-

vide a powerful tool to calculate loop amplitudes in pQCD. To prove its effectiveness we present an example of the one-loop calculation of the five-gluon amplitudes.

The paper is organized as follows. A simple review of the background field method is given in section 2. In section 3, we also review the color decomposition technique and give the color order Feynman rules in the background field gauge. The helicity basis method is a popular technique in multi-jet analysis. We give the review of this method in the section 4. In section 5 we discuss how to combined method is applies to the calculation of the five-gluon amplitudes at one-loop level. The section 6 is the conclusion.

2 Background field gauge

The idea of the background field method is to construct the gauge invariant effective action. In the background field method, the gauge field A in the classical Lagrangian is split into the classical background field B and the quantum field Q.

$$\mathcal{L}(A) = \mathcal{L}(Q + B).$$

For the pure gauge theory, $\mathcal{L}(A) = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ and $F^a_{\mu\nu}$ is the filed strength of A. From the analogy of the conventional generating functional Z,

$$Z[J] = \int \mathcal{D}A \ \det M \exp i\left[\int d^4x\{\mathcal{L}(A) - \frac{1}{2\alpha}G\cdot G + J\cdot A\}\right],$$

we can construct the background field generating functional $\tilde{Z}$ through the orthodox procedure of the path integral formula with the Lagrangian $\mathcal{L}(Q + B)$,

$$\tilde{Z}[J, B] = \int \mathcal{D}Q \ \det \tilde{M} \exp i\left[\int d^4x\{\mathcal{L}(Q + B) - \frac{1}{2\alpha}\tilde{G}\cdot\tilde{G} + J\cdot Q\}\right].$$

Here, G ($\tilde{G}$) term is the gauge fixing term and $\det M$ ($\det \tilde{M}$) is the Faddeev-Popov determinant. Faddeev-Popov determinants are given by the derivative of gauge fixing term under the infinitesimal gauge transformation,

$$\delta A^a_\mu = -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu\omega^a,$$

for $M = \delta G/\delta\omega$, and,

$$\delta Q^a_\mu = -f^{abc}\omega^b (Q + B)^c_\mu + \frac{1}{g}\partial_\mu\omega^a,$$

for $\tilde{M} = \delta\tilde{G}/\delta\omega$. We choose the gauge fixing condition for the background field generating functional as,

$$\tilde{G} = D_\mu^B \cdot Q^{a\mu} \equiv \partial_\mu Q^{a\mu} + g f^{abc} B_\mu^b Q^{c\mu}, \tag{1}$$

which is called the background field gauge. We also get the background field effective action by the Legendre transformation,

$$\tilde{\Gamma}[\tilde{Q}, B] = \tilde{W}[J, B] - \int d^4x J \cdot \tilde{Q},$$

where,

$$\tilde{W}[J, B] = -i \ln \tilde{Z}[J, B] \qquad\qquad \tilde{Q} = \frac{\delta W}{\delta J}.$$

The relation between the background field effective action $\tilde{\Gamma}$ and the conventional effective action Γ is given by,

$$\tilde{\Gamma}[\tilde{Q}, B] = \Gamma[\tilde{Q} + B]_B.$$

The index B in the RHS refers to the B dependence in the gauge fixing term,

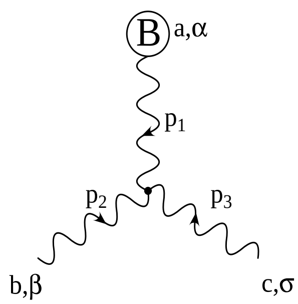
$$G_\mu^a(Q, B) = D_\mu^B \cdot (Q - B)^{a\mu}. \tag{2}$$

For the special case $\tilde{Q} = 0$, we have the important relation[15],

$$\tilde{\Gamma}[0, B] = \Gamma[B]_B.$$

The RHS is the conventional effective action which is calculated with the gauge fixing condition eq.(2). It is invariant under the gauge transformation of B. $\tilde{\Gamma}[0, B]$ has no dependence on the $\tilde{Q}$, thus, 1PI diagrams reduced from this effective action have only background fields B's as the external legs. In other words, the gauge invariant effective action is calculated by summing up the vacuum diagrams in the presence of the classical background field B. Though the background field gauge method provides a different set of Green's functions than conventional methods with conventional gauges, the correct S-matrix is constructed from trees of 1PI diagrams [16].

In addition, the Feynman rules of the background fields gauge have a very simple structure. For example, the three point vertex of the one external and two internal gluons is given as,



$$gf^{abc}\left[(p_2-p_3)_\alpha g_{\beta\sigma}+(p_3-p_1+\frac{1}{\xi}p_2)_\beta g_{\alpha\sigma}\right.$$
$$\left.+(p_1-p_2-\frac{1}{\xi}p_3)_\sigma g_{\alpha\beta}\right].$$

Here, ξ is the gauge fixing parameter for the internal quantum fields. If we choose the gauge fixing parameter as $\xi=1$, the above rule becomes,

$$igf^{abc}[(p_2-p_3)_\alpha g_{\beta\sigma}+2p_{1\sigma}g_{\alpha\beta}-2p_{1\beta}g_{\alpha\sigma}].$$

We notice that only first (p_2-p_3) term includes the internal momenta. Thus, only this term induces the integration momenta into the numerators of the Feynman integrals. In general, loop integrals which include integration momenta in their numerators induce a huge number of terms and make the calculation complicated. Thus, the background field method suppresses the number of the terms which appear in the intermediate stages of the loop calculations remarkably. We will discuss this for the case of the five gluon vertex in the section 5. Note, the same vertex structure also appears in the string inspired method. Bern and Dunbar discussed the relation between the string inspired method and the conventional field theory in ref.[10]. They pointed out that there is a mapping between the string motivated rules for the loop calculation and the Feynman rules of the background field gauge.

3 Color decomposition

The color decomposition is the technique which constructs color ordered gauge invariant subamplitudes in the SU(N) gauge theory. At the tree level, n-point gluon scattering amplitudes $\mathcal{M}_n$ for the SU(N) gauge theory can be decomposed into the subamplitudes m's which are characterized single traces of the group matrices[1]. It is well known as the Chan-Paton factor.

$$\mathcal{M}_n=\sum_{a_i\in S_n/Z_n}\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})m_n(p_{a_1}^{i_{a_1}},p_{a_2}^{i_{a_2}},\cdots,p_{a_n}^{i_{a_n}}),$$

where p_j are external momenta, i_j are helicity. T^a $(a=1,2,\cdots,N^2-1)$ are the matrices of the gauge group in the fundamental representation. S_n/Z_n denotes the set

of noncyclic permutations over $1, \cdots, n$. Each subamplitudes have the independent color structures. Thus these color decomposed subamplitudes are gauge invariant. In addition, $m_n(p_{a_1}^{i_{a_1}}, p_{a_2}^{i_{a_2}}, \cdots, p_{a_n}^{i_{a_n}})$ is invariant under cyclic permutations of $p_j^{i_j}$. The subamplitudes also satisfy some important properties which is known as the Dual Ward identity[1].

For the pure SU(N) gauge theory, we can consider U(N) theory instead of the SU(N) theory. Since the U(1) mode decouples from the SU(N) mode, the U(1) contributions must automatically vanish in the final results of the SU(N) gluon amplitudes. However, U(N) group has the larger symmetry than the SU(N) group. Thus presence of the U(1) mode simplify the intermediate stage of the calculations. T^a $(a = 1, 2, \cdots, N^2 - 1)$ are generators of SU(N) group for the fundamental representation. In this paper, we use the normalization condition of the generators as $\mathrm{Tr}(T^a T^b) = \delta^{ab}/2$. (This normalization is different from the Bern et.al.'s one in the factor $\sqrt{2}$.) It satisfies the relation between the adjoint representation and the fundamental representation,

$$\mathrm{Tr}[T^a\ ,[T^b, T^c]] = \frac{i}{2} f^{abc}.$$

We introduce the U(1) mode T^0,

$$T^0_{ij} = \frac{\delta_{ij}}{\sqrt{2N}}.$$

The factor $\frac{1}{\sqrt{2N}}$ is convention which does not change the normalization condition of T's. The algebra is modified as following from SU(N) to SU(N)$\times$ U(1) theory.

$$\sum_{a=1}^{N^2-1} T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \quad \Longrightarrow \quad \sum_{a=0}^{N^2-1} T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk}\right).$$

Here we sum up over the indices $c = 0, 1, 2, \cdots, N^2 - 1$ for the SU(N)$\times$ U(1) . The Casimir factor is given by,

$$\sum_{a=1}^{N^2-1} T^a T^a = \frac{N^2 - 1}{2N} \quad \Longrightarrow \quad \sum_{a=0}^{N^2-1} T^a T^a = \frac{N}{2}$$

We also obtain the following simple formulas of the Fierz identities,

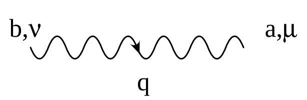
$$\sum_{b=0}^{N^2-1} Tr(T^{a_1} T^{a_2} \cdots T^{a_m} T^b)(T^b\ T^{a_{m+1}} \cdots T^{a_n}) \;=\; \frac{1}{2} Tr(T^{a_1} T^{a_2} \cdots T^{a_m} T^{a_{m+1}} \cdots T^{a_n}),$$

$$\sum_{b=0}^{N^2-1} Tr(T^{a_1}\cdots T^{a_m}T^b\ T^{a_{m+1}}\cdots T^{a_n}\ T^b) \ = \ \frac{1}{2}Tr(T^{a_1}\cdots T^{a_m})Tr(T^{a_{m+1}}\cdots T^{a_n}). \tag{3}$$

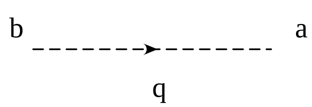
For the one loop level gluon amplitudes, double-trace components also appear. For example, the color decomposition of the one-loop five-gluon amplitude is given by[11],

$$\begin{aligned} \mathcal{M}_n \ &= \sum_{a_i\in S_5/Z_5} \mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_5})m_{5,1}(p_{a_1}^{i_{a_1}},\cdots,p_{a_5}^{i_{a_5}}) \\ &+ \sum_{a_i\in S_5/S_{5;2}} \mathrm{Tr}(T^{a_1}T^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4}T^{a_5})m_{5,3}(p_{a_1}^{i_{a_1}},\cdots,p_{a_5}^{i_{a_5}}) \end{aligned}$$

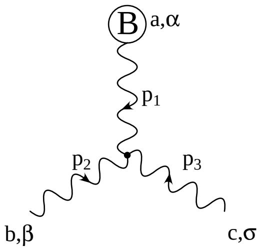
Double-trace components $m_{5,3}$ are related with $m_{5,1}$ via the decoupling equation. This relation can be easily derived from the string theory[11]. We also can derive it in the straightforward way with using the U(N) Fierz identities eq.(3). So, we only need to consider the single-trace part $m_{5,1}$. Single-trace parts $m_{5,1}$ are a leading contribution of the large N expansion in the U(N) and SU(N) gauge theory. Leading order contributions of the large N expansion are given directly with using the color ordered Feynman rules. In addition, only a color ordered subset of all the Feynman diagrams is required. In other word, we only have to consider the topologically independent diagrams and their cyclic permutations on the external color charges to calculate the amplitudes $m_{5,1}$. Thus we reach the color-ordered Feynman rules of the background field gauge as summarized in the following. The color factors $(a, b, \cdots)$ denote $\mathrm{Tr}(T^a, T^b, \cdots)$. We also give the diagrams of the five gluon vertices at one-loop level in Appendix.



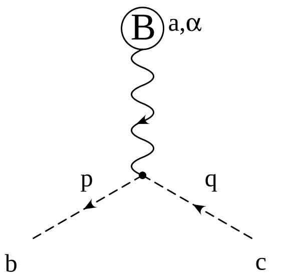
$$-\frac{i\delta_{ab}g_{\mu\nu}}{q^2+i\varepsilon}$$



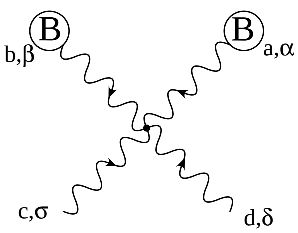
$$\frac{i\delta_{ab}}{q^2+i\varepsilon}$$



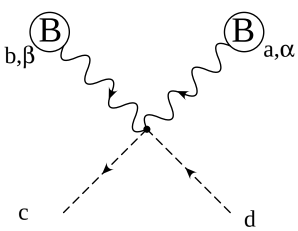
$$-2ig\ (abc)[(p_2-p_3)_\alpha g_{\beta\sigma}+2p_{1\sigma}g_{\alpha\beta}-2p_{1\beta}g_{\alpha\sigma}]$$



$$-2ig\ (abc)(p+q)_\alpha$$



$$-4ig^2(abcd)(g_{\alpha\delta}g_{\beta\sigma}-g_{\alpha\sigma}g_{\beta\delta}+\frac{1}{2}g_{\alpha\beta}g_{\sigma\delta})$$



$$2ig^2\ (abcd)g_{\alpha\beta}$$

4 Spinor Helicity basis

It is well known that the helicity basis method is often useful in the tree level calculations[1, 3]. It is useful in the loop calculation, too[17]. In this method, we calculate matrix elements in which polarizations of the external fields are characterized by the spinor helicity basis. Here we introduce the well known notations on the helicity basis of a spinor field ψ as,

$$\langle q^\pm| = \frac{1}{2}\overline{\psi}(q)(1\mp\gamma_5), \qquad |\bar{q}^\pm\rangle = \frac{1}{2}(1\pm\gamma_5)\psi(\bar{q}).$$

Here all fermions are massless. The normalization condition is given by,

$$\langle p|\gamma_\mu|p\rangle = 2p_\mu.$$

It is convenient to introduce the following notation of spinor products,

$$\langle pq\rangle = \langle p^-|q^+\rangle, \qquad [pq] = \langle p^+|q^-\rangle, \qquad [pq]\langle qp\rangle = s_{pq} = 2p\cdot q.$$

The most remarkable advantage of this formula is that we can also describe external gauge fields by using the spinor helicity basis. Thus, the complicated tensor structure of the multi-gluon amplitudes are replaced into the calculation of the Dirac algebra. The polarization vectors may be written in terms of massless spinors $|p^\pm\rangle$ and $|k^\pm\rangle$,

$$\varepsilon^\pm(p,k) = \pm\frac{\langle p^\pm|\gamma_\mu|k^\pm\rangle}{\sqrt{2}\langle k^\mp|p^\pm\rangle} \tag{4}$$

where p is the gauge boson momentum, k is the arbitrary momentum which satisfies $k^2 = 0$. We call this momentum k as the *reference* momentum.

The final results for physical observable do not depend on the *reference* momentum because a change in the reference momentum is equivalent to a gauge transformation:

$$\varepsilon^+(p,k)_\mu \to \varepsilon^+(p,k')_\mu - \sqrt{2}\frac{\langle kk'\rangle}{\langle kp\rangle\langle k'p\rangle}p_\mu.$$

This means we have freedom in choosing an appropriate reference momentum for any gauge invariant subset of the full amplitude, such as a gauge invariant color-ordered subamplitude.

The polarization vectors defined by eq.(4) satisfy not only the equation of motion,

$$p_\mu\varepsilon^\pm(p,k)^\mu = 0,$$

but also,

$$k_\mu\varepsilon^\pm(p,k)^\mu = 0.$$

Using the Fierz identity,

$$\langle p^+|\gamma_\mu|q^+\rangle\langle k^-|\gamma^\mu|l^-\rangle = 2[pl]\langle kq\rangle, \tag{5}$$

and a symmetric property,

$$\langle p^+|\gamma_\mu|q^+\rangle = \langle q^-|\gamma_\mu|p^-\rangle,$$

it is easy to show the following relations,

$$\begin{aligned}\varepsilon^{\pm}(p,k)\cdot\varepsilon^{\pm}(p,k') &= 0\\ \varepsilon^{\pm}(p,k)\cdot\varepsilon^{\pm}(p',k) &= 0.\end{aligned}$$

In the actual calculations, we take advantage of the fact that choices of the reference momenta are not unique. Some of the good choices of the reference momenta make it easy to use the above identities which reduce the number of terms in the calculations. The case of the four gluon amplitudes are shown in the ref.[6].

For the five gluon case, in this paper, we choose the reference momenta of the helicity + gluons in the $m_5(+,+,+,+,+)$ amplitude as,

$$\epsilon_\mu(l_i,k_i=l_{i+1})=\frac{\langle l_i|\gamma_\mu|l_{i+1}\rangle}{\sqrt{2}\langle l_{i+1}l_i\rangle},$$

where l_i (k_i) is a i-th external gluon (reference) momentum. Using this expression, we can replace the complicated tensor structures into the calculation of the Dirac algebra. For example, contraction of the momentum l's and helicity + external gluon fields are,

$$\begin{aligned}l_i^\alpha l_j^\beta l_k^\sigma l_l^\delta l_m^\rho\times\varepsilon_\alpha^{1+}\varepsilon_\beta^{2+}\varepsilon_\sigma^{3+}\varepsilon_\delta^{4+}\varepsilon_\rho^{5+} &= \frac{\langle 1^+|i|2^+\rangle\langle 2^+|j|3^+\rangle\langle 3^+|k|4^+\rangle\langle 4^+|l|5^+\rangle\langle 5^+|m|1^+\rangle}{(\sqrt{2})^5\langle 21\rangle\langle 32\rangle\langle 43\rangle\langle 54\rangle\langle 15\rangle}\\ &= \frac{Tr(1i2j3k4l5mP_+)}{(\sqrt{2})^5\langle 21\rangle\langle 32\rangle\langle 43\rangle\langle 54\rangle\langle 15\rangle},\end{aligned}$$

where $\varepsilon_\mu^{i+}\equiv\varepsilon_\mu^+(l_i,k=l_{i+1})$, $\langle i|j|k\rangle\equiv\langle l_i|\not{l}_j|l_k\rangle$ and $\langle ij\rangle\equiv\langle l_il_j\rangle$, $P_+\equiv\frac{1}{2}(1+\gamma_5)$, $Tr(ij\cdots)\equiv Tr(\not{l}_i\not{l}_j\cdots)$. Here we used the identities,

$$\langle p|k|q\rangle=[pk]\langle kq\rangle \tag{6}$$

and

$$\begin{aligned}\frac{1}{2}Tr(\not{p}_1\not{p}_2\cdots\not{p}_{2n}(1+\gamma_5)) &= [p_1p_2]\langle p_2p_3\rangle\cdots\langle p_{2n}p_1\rangle\\ \frac{1}{2}Tr(\not{p}_1\not{p}_2\cdots\not{p}_{2n}(1-\gamma_5)) &= \langle p_1p_2\rangle[p_2p_3]\cdots[p_{2n}p_1].\end{aligned} \tag{7}$$

We also choose $(k_1,k_2,k_3,k_4,k_5)=(l_2,l_3,l_4,l_5,l_2)$ for the $m_5(-,+,+,+,+)$ case. Of curse, the choice of the reference momentum is not unique. Other choice is possible and may be more efficient efficient.

In this paper, we would like to apply the helicity basis method to the one-loop calculation. To carry out the Feynman integral, the dimensional regularization is efficient. In the conventional dimensional regularization scheme, all gluon polarizations are dealt with in the $4-2\varepsilon$ dimension. On the other hand, in the helicity basis method, gluon polarization vectors are defined in 4 dimensions, because, the spinor helicity basis is only well defined in 4 dimensions. Thus, we need some modification on the regularization scheme. The Four Dimensional Helicity(FDH) scheme is one of the solutions which is effective in the helicity basis method[6]. In this scheme, all gluon polarizations (of observed and unobserved) are dealt with in 4 dimensions. Thus all gluons have 2 helicity states. The 'tHooft-Veltman scheme is also applicable. But, in this scheme, unobserved gluon (virtual, soft and collinear) polarizations are kept in $4-2\varepsilon$ dimensions and we only treat the observed gluon polarizations in 4 dimensions.

Before discussing the one-loop calculation in the next section, we give some comments on the tree-level results. For the tree level calculation, supersymmetry is useful[4]. Supersymmetric Ward identities show that maximal helicity violating(MHV) and next helicity violating(NHV) multi-gluon amplitudes of the pure supersymmetric QCD vanish,

$$m_n^{SUSY}(l_1^{\pm}, l_2^{+}, \cdots, l_n^{+}) = 0,$$

where l_i are external momentums and indices $\pm$ denote helicities. In the supersymmetric theory, additional super particles, eg. scalar gluons and gluinos, contribute to the amplitudes. However, if we assume the R-symmetry, tree level amplitudes include no exotic couplings between gluons and additional particles. Thus, the non-supersymmetric MHV and NHV amplitudes for general n gluon vertices also vanish in the tree level,

$$m_n^{tree}(l_1^{\pm}, l_2^{+}, \cdots, l_n^{+}) = 0.$$

This simple results ensure that MHV and NHV amplitudes in the one loop level must be infrared finite.

For the other combinations of the helicities, amplitudes do not vanish. But using the spinor helicity formula, we obtain very simple expressions of the color ordered helicity amplitudes,

$$m_n^{tree}(p_1^{+}, p_2^{+}, \cdots, p_i^{-}, \cdots, p_j^{-}, \cdots, p_n^{+}) = ig^{n-2}(\sqrt{2})^n \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle},$$

where $\langle ij\rangle = \langle p_i p_j\rangle$. One-loop amplitudes for these helicities induce the Infrared singularity[7].

5 Five gluon vertex example

In this section, we would like to demonstrate the background field gauge combined with the helicity basis method and the color decomposition is a powerful tool for the one-loop calculation. We perform the one-loop calculation of the five gluon amplitudes as an example. The one-loop level five gluon amplitudes were first calculated by Bern, Dixon and Kosower with using the new technique which is called the string inspired method. Here we show that the background field gauge is powerful, too. Combination of the background field gauge, the color decomposition and the helicity basis method simplifies the calculation enough and make it feasible to compute the one-loop five gluon amplitudes in the straightforward way.

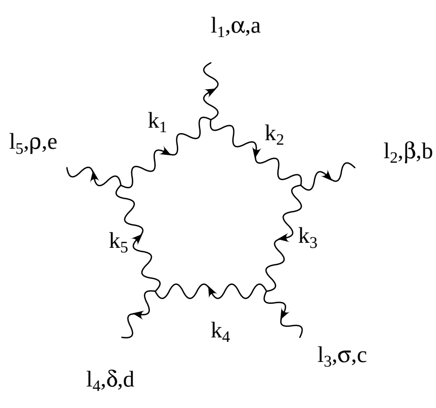
To carry out this calculation, we need the information of the tensor type Feynman integrals for the pentagon diagram. We follow the technology on the Feynman integral calculation which discussed in references [18]. Here we consider the general form of the dimensionally regulated massless Pentagon integral,

$$\mathcal{I}_5[k^\mu k^\nu k^\rho\cdots] \equiv \int \frac{d^Dk}{(2\pi)^D}\frac{\mu^{2\epsilon}k^\mu k^\nu k^\rho\cdots}{k^2(k-p_1)^k(k-p_2)^2(k-p_3)^2(k-p_4)^2}, \tag{8}$$

where $D = 4-2\varepsilon$, μ is the dimensional regulation scale parameter, $p_i = \sum_{j=1}^{i} l_j$ and l_i are external momenta. We also introduce the following notation to simplify the calculation of the tensor integrals,

$$I_5[k^\mu k^\nu k^\rho\cdots] \equiv i(4\pi)^{2-\epsilon}\mu^{-2\epsilon}\mathcal{I}_5[k^\mu k^\nu k^\rho\cdots]. \tag{9}$$

For example, using the color ordered Feynman rules presented in section 3, explicit form of the color ordered gluonic pentagon integral, is given by,



$$\begin{aligned} I_5^g &= i^5 N\ Tr(abcde) \\ &\times \int \frac{d^4k}{(2\pi)^4}\frac{(-i2g)^5}{k_1^2k_2^2k_3^2k_4^2k_5^2} \\ &\times [k_{1\alpha}g_{\mu\nu} + l_{1\mu}g_{\alpha\nu} - l_{1\nu}g_{\alpha\mu}] \\ &\times [k_{2\beta}g_{\nu\lambda} + l_{2\nu}g_{\beta\lambda} - l_{2\lambda}g_{\beta\nu}] \\ &\times [k_{3\sigma}g_{\tau\lambda} + l_{3\lambda}g_{\sigma\tau} - l_{3\tau}g_{\sigma\lambda}] \\ &\times [k_{4\delta}g_{\xi\tau} + l_{4\tau}g_{\delta\xi} - l_{4\xi}g_{\delta\tau}] \\ &\times [k_{5\rho}g_{\mu\xi} + l_{5\xi}g_{\rho\mu} - l_{5\mu}g_{\rho\xi}]. \end{aligned}$$

where $Tr(ab\cdots) = Tr(T^aT^b\cdots)$, k_i are internal momenta and l_i are external gluon

momentums. We notice that only the first term of the each vertices include the integration momenta k. Thus, the amplitude I_5^g is decomposed into

$$\begin{aligned} I_5^g &= F_0 I_5[k_1^\alpha k_2^\beta k_3^\sigma k_4^\delta k_5^\rho] + F_1^\rho I_5[k_1^\alpha k_2^\beta k_3^\sigma k_4^\delta] + F_2^{(1)\delta\rho} I_5[k_1^\alpha k_2^\beta k_3^\sigma] + F_2^{(2)\sigma\rho} I_5[k_1^\alpha k_2^\beta k_4^\delta] \\ &+ F_3^{(1)\sigma\delta\rho} I_5[k_1^\alpha k_2^\beta] + F_3^{(2)\beta\delta\rho} I_5[k_1^\alpha k_3^\sigma] + F_4^{\beta\sigma\delta\rho} I_5[k_1^\alpha] + F_5^{\alpha\beta\sigma\delta\rho} I_5[1] \\ &+ \text{cyclic permutation} \end{aligned}$$

If we use the conventional gauge, like the covariant gauge, other terms also include the integration momenta k. Thus, more combinations on $I_5[k's]$ appear.

Integrating over the momentum k after the Feynman parameterization, the momentum integral (9) is rewritten by the Feynman parameter integral. The tensor structure is decomposed by the terms of momentums $l's$ and metric tensors $g_{\mu\nu}$. For example, the tensor integrals which appear in the pentagon integral I_5^g, is described by,

$$\begin{aligned} I_5[k^{\mu_1}] &= \sum_{i=1}^{4} p_i^{\mu_1} I_5[a_{i+1}] \\ I_5[k^{\mu_1} k^{\mu_2}] &= \sum_{i,j=1}^{4} p_i^{\mu_1} p_j^{\mu_2} I_5[a_{i+1} a_{j+1}] - \frac{1}{2} g^{\mu_1\mu_2} I_5^{D=6-2\epsilon} \\ I_5[k^{\mu_1} k^{\mu_2} k^{\mu_3}] &= \sum_{i,j,k=1}^{4} p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} I_5[a_{i+1} a_{j+1} a_{k+1}] - \frac{1}{2} \sum_i \{g p_i\}^{\mu_1\mu_2\mu_3} I_5^{D=6-2\epsilon}[a_{i+1}], \\ I_5[k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4}] &= \sum_{i,j,k,l=1}^{4} p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} I_5[a_{i+1} a_{j+1} a_{k+1} a_{l+1}] \\ &\quad - \frac{1}{2} \sum_{i,j} \{g p_i p_j\}^{\mu_1\mu_2\mu_3\mu_4} I_5^{D=6-2\epsilon}[a_{i+1} a_{j+1}] + \frac{1}{4} \{gg\}^{\mu_1\mu_2\mu_3\mu_4} I_5^{D=8-2\epsilon}, \\ I_5[k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4} k^{\mu_5}] &= \sum_{i,j,k,l,m=1}^{4} p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} p_m^{\mu_5} I_5[a_{i+1} a_{j+1} a_{k+1} a_{l+1} a_{m+1}] \\ &\quad - \frac{1}{2} \sum_{i,j,k} \{g p_i p_j p_k\}^{\mu_1\mu_2\mu_3\mu_4\mu_5} I_5^{D=6-2\epsilon}[a_{i+1} a_{j+1} a_{k+1}] \\ &\quad + \frac{1}{4} \sum_i \{g g p_i\}^{\mu_1\mu_2\mu_3\mu_4\mu_5} I_5^{D=8-2\epsilon}[a_{i+1}], \end{aligned} \tag{10}$$

where $p_i = \sum_{j=1}^{i} l_j$, $I_5[a_i, \cdots]$ are Feynman parameter integrals which are same as

the notation introduced in ref.[18] and a_i are Feynman parameters,

$$I_n[a_{i_1}\cdots a_{i_m}] \equiv \Gamma(n-2+\epsilon)\int_0^1 d^n a_i \delta(1-\sum_i a_a)\frac{a_{i_1}\cdots a_{i_m}}{\mathcal{D}(a_i)^{n-2+\epsilon}}$$

$$\begin{aligned}\mathcal{D}(a_i) &\equiv \sum_{i.j=1}^{n} D_{ij}a_i a_j. \qquad\qquad D_{ij}\equiv\frac{1}{2}(-P_{ij}^2)\\ P_{ij} &\equiv p_{j-1}-p_{i-1} = l_i + l_{i+1}+\cdots+l_{j-1} \quad \text{for} i<j. \end{aligned}\tag{11}$$

The $\{gp_i\cdots\}^{\mu_1\cdots}$ denotes the summation over the all possible permutations of Lorentz indices, for example,

$$\{gp_i\}^{\mu_1\mu_2\mu_3} = g^{\mu_1\mu_2}p_i^{\mu_3} + g^{\mu_3\mu_1}p_i^{\mu_2} + g^{\mu_2\mu_3}p_i^{\mu_1}.$$

To perform the Feynman parameter integrals, we use the dimensional regulated formula discussed by Bern et.al.[18]. The idea of this formula is to construct algebraic equations for n-point one-loop integrals. For the $D=4-2\epsilon$ scalar n-point integral case, we have,

$$I_n[a_i] = \frac{1}{2}\left\{\sum_{j=1}^{n} D_{ij}^{-1} I_{n-1}^{(i)}[1] + (n-5+2\epsilon)c_i I_n^{D=6-2\epsilon}[1]\right\},\tag{12}$$

where,

$$c_i = \sum_{j=1}^{n} D_{ij}^{-1},$$

and D_{ij} is defined in eq.(11). $I_{n-1}^{(k)}$ is the n-1 point integral corresponds to removing the propagater parameterized by a_k from the integral I_n. Using the identity $\sum_i^n I_n[a_i] = I_n[1]$, we have,

$$I_n[1] = \frac{1}{2}\left\{\sum_{i=1}^{n} c_i I_{n-1}^{(i)}[1] + (n-5+2\epsilon)c_0 I_n^{D=6-2\epsilon}[1]\right\},\tag{13}$$

where, $c_0=\sum_{i=1}^n c_i$. Since $I_5^{D=6-2\epsilon}$ is finite, $\epsilon\to 0$ limit for $n=5$ case reproduces the Melrose and van Neerven et.al.'s result[20],

$$I_5[1] = \frac{1}{2}\sum_{i=1}^{5} c_i I_4^{(i)}[1] + \mathcal{O}(\epsilon).$$

We notice that the scalar pentagon integral in the 4 dimension is obtained by a sum of the five box integrals. For the calculation of the tensor type Feynman integral, we also need the information of the Feynman integrals $I_n[a_i, a_j \cdots]$. By the changing of the integration variables in eq.(11) from a_i to u_i[21],

$$a_i = \frac{\alpha_i u_i}{\sum_{j=1}^{n} \alpha_j u_j}, \qquad a_n = \frac{\alpha_n (1 - \sum_{j=1}^{n-1} u_j)}{\sum_{j=1}^{n} \alpha_j u_j},$$

it is very easy to show,

$$I_n[a_{i_1} a_{i_2} \cdots a_{i_m}] = \frac{\Gamma(n-3-m+2\epsilon)}{\Gamma(n-3+2\epsilon)} \mathcal{A}_n \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_m} \frac{\partial}{\partial \alpha_{i_1}} \frac{\partial}{\partial \alpha_{i_2}} \cdots \frac{\partial}{\partial \alpha_{i_m}} \left(\frac{I_n[1]}{\mathcal{A}_n} \right), \tag{14}$$

where $\mathcal{A}_n = \prod_{j=1}^{n} \alpha_j$. From the eq.(12) and eq.(13),the scalar pentagon integral $I_5[1]$ and the one parameter pentagon integral $I_5[a_i]$ are described by $D = 4 - 2\epsilon$ box integrals in the $\mathcal{O}(\epsilon^0)$. Explicit form of the scalar box integral is,

$$I_4^{(k)} = 2\gamma_\Gamma \mathcal{A}_4^{(k)} \left[\frac{(\alpha_{k+2}\alpha_{k-2})^\epsilon}{\epsilon^2} + \mathrm{Li}_2\left(1 - \frac{\alpha_{k+1}}{\alpha_{k+2}}\right) + \mathrm{Li}_2\left(1 - \frac{\alpha_{k-1}}{\alpha_{k-2}}\right) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon),$$

where $\mathcal{A}_4^{(k)} = \prod_{j(\neq k)=1}^{4} \alpha_j$. $\mathrm{Li}_2(Z)$ is the Spence function,

$$\mathrm{Li}_2(Z) = -\int_0^Z \frac{dx}{x} \log(1-x).$$

The parameters α_i satisfy $s_{ii+1} = 1/\alpha_i \alpha_{i+2}$. Thus, pentagon integrals $I_5[1]$ and $I_5[a_i]$ are given as,

$$\begin{aligned} I_5[1] &= \gamma_\Gamma \sum_{j=1}^{5} \alpha_j^{1+2\epsilon} A_5 \left[\frac{1}{\epsilon^2} + 2\mathrm{Li}_2\left(1 - \frac{\alpha_{j+1}}{\alpha_j}\right) + 2\mathrm{Li}_2\left(1 - \frac{\alpha_{j-1}}{\alpha_j}\right) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon) \\ I_5[a_i] &= \gamma_\Gamma A_5 \alpha_i \left[\frac{\alpha_i^{2\epsilon}}{\epsilon^2} + 2\mathrm{Li}_2\left(1 - \frac{\alpha_{j+1}}{\alpha_j}\right) + 2\mathrm{Li}_2\left(1 - \frac{\alpha_{j-1}}{\alpha_j}\right) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon) \end{aligned} \tag{15}$$

The pentagon integrals which have two Feynman parameters in the numerator is calculated by the following relation,

$$I_5^{D=4-2\epsilon}[a_i a_j] = \frac{\alpha_i \gamma_i}{\hat{\Delta}_n} I_5^{D=4-2\epsilon}[a_j] + \sum_k \frac{\mathcal{R}_{ik}}{2} \overset{(k)}{I_4^{D=4-2\epsilon}}[a_j] + \frac{\mathcal{R}_{ij}}{2} I_5^{D=6-2\epsilon}[1]. \tag{16}$$

where,

$$
\begin{aligned}
\hat{\Delta}_5 &= \sum_{i=1}^{5}(\alpha_i^2 - 2\alpha_i\alpha_{i+1} + 2\alpha_i\alpha_{i+2}) = \gamma_5\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_5 + \gamma_4\gamma_1 \\
\gamma_i &= \frac{1}{2}\frac{\partial\hat{\Delta}_5}{\partial\alpha_i} = \alpha_{i-2} - \alpha_{i-1} + \alpha_i - \alpha_{i+1} + \alpha_{i+2}, \\
\mathcal{R}_{ij} &= \alpha_i\alpha_j(\eta_{ij} - \frac{\gamma_i\gamma_j}{\hat{\Delta}_5}) \qquad\qquad \eta_{ij} = \begin{cases} -1 & (i = j \pm 1) \\ +1 & (\text{others}) \end{cases}
\end{aligned}
$$

The box integral $I_4^{(k)}[a_i]$ in eq.(16) is,

$$
\begin{aligned}
I_4^{\overset{(k)}{D=4-2\epsilon}}[a_i] &= \gamma_\Gamma \mathcal{A}_4^{(k)}\alpha_i \left[\delta_{k,i-1}\left\{-\frac{1}{\epsilon^2}\frac{\alpha_{i-2}^\epsilon(\alpha_i^\epsilon - \alpha_{i+1}^\epsilon)}{\alpha_i - \alpha_{i+1}} + \frac{\alpha_{i+2}L_k}{\hat{\Delta}_5 - \gamma_k^2}\right\}\right. \\
&\quad +\delta_{k,i-2}\left\{\frac{1}{\epsilon^2}\left(\frac{\alpha_{i-1}^\epsilon\alpha_{i+1}^\epsilon}{\alpha_i} + \frac{\alpha_{i+2}^\epsilon(\alpha_{i-1}^\epsilon - \alpha_i^\epsilon)}{\alpha_{i-1} - \alpha_i}\right) + \frac{(\alpha_{i+2} - \alpha_{i+1})L_k}{\hat{\Delta}_5 - \gamma_k^2}\right\} \\
&\quad +\delta_{k,i+2}\left\{\frac{1}{\epsilon^2}\left(\frac{\alpha_{i-1}^\epsilon\alpha_{i+1}^\epsilon}{\alpha_i} + \frac{\alpha_{i-2}^\epsilon(\alpha_{i+1}^\epsilon - \alpha_i^\epsilon)}{\alpha_{i+1} - \alpha_i}\right) + \frac{(\alpha_{i-2} - \alpha_{i-1})L_k}{\hat{\Delta}_5 - \gamma_k^2}\right\} \\
&\quad \left. + \delta_{k,i+1}\left\{-\frac{1}{\epsilon^2}\frac{\alpha_{i+2}^\epsilon(\alpha_i^\epsilon - \alpha_{i-1}^\epsilon)}{\alpha_i - \alpha_{i-1}} + \frac{\alpha_{i-2}L_k}{\hat{\Delta}_5 - \gamma_k^2}\right\}\right],
\end{aligned}
$$

where

$$
L_i = Li_2(1 - \frac{\alpha_{i+1}}{\alpha_{i+2}}) + Li_2(1 - \frac{\alpha_{i-1}}{\alpha_{i-2}}) + \ln\frac{\alpha_{i+1}}{\alpha_{i+2}}\ln\frac{\alpha_{i-1}}{\alpha_{i-2}} - \frac{1}{6}\pi^2.
$$

$D = 6-2\epsilon$ integrals are given from the analytic continuation $D = 4-2\epsilon$ to $D = 6-2\epsilon$ by the shift $\epsilon \to \epsilon-1$. But as is well known that the coefficient of scalar integral $I_5^{D=6}$ always vanishes from the tensor integrals[18]. Thus we do not have to consider this contribution in the actual calculation. Using these results of the box integrals and eq.(12)-(14) we calculate the pentagon integrals $I_5[a_i,\cdots]$ and other box integrals $I_4^{(k)}[a_ia_j\cdots]$. To perform all Feynman integrals automatically, we made the program of the Maple[22].

Now we come back to the calculation of the five gluon vertex. First we consider typical helicities case $m_5(+,+,+,+,+)$ as the simplest example. In this case, since the tree level amplitude vanishes the one-loop amplitude is infrared finite. We have variety of choices of the reference momentums for the external gluon fields. We choose a reference momentum k_i of a i-th gluon with a momentum l_i as,

$$
\epsilon_\mu(l_i, k_i = l_{i+1}) = \frac{\langle l_i|\gamma_\mu|l_{i+1}\rangle}{\sqrt{2}\langle l_{i+1}l_i\rangle}, \qquad (17)
$$

where l_{i+1} is a $(i+1)$-th gluon momentum. Using the identities,

$$l_i^{\mu_i}\varepsilon^+_{\mu_i}(l_i, k = l_{i+1}) = l_{i+1}^{\mu_i}\varepsilon^+_{\mu_i}(l_i, k = l_{i+1}) = 0,$$

the tensor integrals can be replaced as following ,

$$\begin{aligned} I_5[k_1^\alpha k_2^\beta k_3^\sigma k_4^\delta \cdots] &= i\int \frac{d^D k}{\pi^{D/2}} \frac{k_1^\alpha k_2^\beta k_3^\sigma k_4^\delta \cdots}{k_1^2 k_2^2 k_3^2 k_4^2 k_5^2} \\ &= \sum_{i,j,k,cdots=1}^{4} \tilde{p}_i^\alpha \tilde{p}_j^\beta \tilde{p}_k^\delta \tilde{p}_l^\sigma \cdots I_5[a_{i+1}a_{j+1}a_{k+1}a_{l+1}\cdots] \\ &- \frac{1}{2}\sum_{ij\cdots}\{g\tilde{p}_i\tilde{p}_j\cdots\}^{\alpha\beta\sigma\delta\cdots} I_5^{D=6-2\epsilon}[a_{i+1}a_{j+1}\cdots] \\ &+ \frac{1}{4}\{gg\cdots\}^{\alpha\beta\sigma\delta\cdots} I_5^{D=8-2\varepsilon}[\cdots] \end{aligned} \tag{18}$$

here we ignore the terms which disappear by contracting gluon polarization vectors eq.(17). $\tilde{p}_i$ are given in the table.1. Other combinations on k's are also given by the permutations of $\tilde{p}$.

Table.1

i	1	2	3	4	5
$\tilde{p}_i^\alpha$	0	0	l_3^α	$-l_5^\alpha$	0
$\tilde{p}_i^\beta$	0	0	0	l_4^β	$-l_1^\beta$
$\tilde{p}_i^\sigma$	$-l_2^\sigma$	0	0	0	l_5^σ
$\tilde{p}_i^\delta$	l_1^δ	$-l_3^\delta$	0	0	0
$\tilde{p}_i^\rho$	0	l_2^ρ	$-l_4^\rho$	0	0

In addition, from the identities eq.(6)$\sim$ (7) and the expression eq.(17) of the external gluons, tensor structures are replaced into traces of the γ matrices. For example, contraction of the momentum l's and metric tensor $g_{\mu\nu}$ with external gluon fields are,

$$\begin{aligned} l_i^\alpha l_j^\beta l_k^\sigma l_l^\delta l_m^\rho \times \varepsilon_\alpha^{1+}\varepsilon_\beta^{2+}\varepsilon_\sigma^{3+}\varepsilon_\delta^{4+}\varepsilon_\rho^{5+} &= \frac{Tr(1i2j3k4l5mP_+)}{(\sqrt{2})^5\langle 21\rangle\langle 32\rangle\langle 43\rangle\langle 54\rangle\langle 15\rangle} \\ l_i^\alpha l_j^\beta l_k^\sigma g^{\delta\rho} \times \varepsilon_\alpha^{1+}\varepsilon_\beta^{2+}\varepsilon_\sigma^{3+}\varepsilon_\delta^{4+}\varepsilon_\rho^{5+} &= \frac{-2Tr(1i2j3k45P_+)}{(\sqrt{2})^5\langle 21\rangle\langle 32\rangle\langle 43\rangle\langle 54\rangle\langle 15\rangle} \\ &\cdots \quad , \end{aligned}$$

where $\varepsilon_\mu^{i+} = \varepsilon_\mu^+(l_i, k = l_{i+1})$, $P_+ \equiv \frac{1}{2}(1+\gamma_5)$, $Tr(ij\cdots) \equiv Tr(\not{l}_i\ \not{l}_j\cdots)$. To calculate these Dirac algebra, we used the algebraic manipulation program FORM.

One-loop Feynman diagrams which contribute to the m_5 are given in appendix. Summing up all contributions of these diagrams, we have the following simple final result of the color ordered MHV amplitude $m_{5;1}(+,+,+,+,+)$,

$$m_{5;1}(+,+,+,+,+) = \frac{i(\sqrt{2})^5 N}{96\pi^2} \frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + \epsilon(1,2,3,4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

where $s_{ij} = 2l_i \cdot l_j$ and $\epsilon(ijkm) = i4\epsilon_{\mu\nu\rho\sigma} l_i^\mu l_j^\nu l_k^\rho l_m^\sigma$. We also applied this program to the NHV amplitude. For the NHV amplitude $m_{5;1}(-,+,+,+,+)$, we chose the reference momenta k_i as $(k_1,k_2,k_3,k_4,k_5) = (l_2,l_3,l_4,l_5,l_2)$. The NHV amplitude $m_{5;1}(-,+,+,+,+)$ is given by,

$$m_{5;1}(-,+,+,+,+) = \frac{i(\sqrt{2})^5 N}{96\pi^2} \frac{A + B \times \epsilon(1,2,3,4)}{[12][51]\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 25\rangle^2},$$

where

$$\begin{aligned}
A &= (s_{34}+s_{23})\,\frac{s_{12}}{s_{15}}\left\{\frac{{s_{45}}^2 s_{34}}{s_{23}} + \frac{{s_{12}}^2 s_{23}}{s_{34}} + 2\,s_{12}s_{45} + s_{23}s_{34} - 2\,s_{34}s_{45} - 2\,s_{23}s_{12}\right\}\\
&+ (s_{34}+s_{45})\,\frac{s_{15}}{s_{12}}\left\{\frac{{s_{23}}^2 S_{34}}{s_{45}} + \frac{{s_{15}}^2 s_{45}}{s_{34}} + 2\,s_{15}s_{23} + s_{34}s_{45} - 2\,s_{23}s_{34} - 2\,s_{15}s_{45}\right\}\\
&- \frac{s_{12}s_{13}}{s_{23}}\,(s_{15}s_{12} + 2\,s_{34}s_{45} - s_{15}s_{45}) - \frac{s_{15}s_{14}}{s_{45}}\,(2\,s_{23}s_{34} - s_{23}s_{12} + s_{15}s_{12})\\
&+ \frac{1}{s_{34}}\{(s_{15}+s_{12})^2(s_{15}s_{12} + s_{23}s_{45} - 2s_{12}s_{23} - 2s_{15}s_{45})\\
&\quad +(s_{15}s_{45} + s_{15}s_{23} + s_{12}s_{45} + s_{12}s_{23})(s_{15}s_{45} + s_{12}s_{23}) + s_{15}^2 s_{45}^2 + s_{12}^2 s_{23}^2\}\\
&- 6s_{15}s_{34}s_{45} + 2s_{15}s_{23}s_{45} + s_{15}s_{34}s_{23} + 2s_{12}s_{23}s_{45} + s_{15}s_{12}s_{34} + 2s_{15}^2 s_{23}\\
&+ 2s_{12}^2 s_{45} + s_{34}^2 s_{45} + s_{23}s_{34}^2 - 2s_{12}s_{23}^2 + 7s_{23}s_{12}^2 - 2s_{15}s_{45}^2 + 7s_{15}^2 s_{45}\\
&- 6s_{23}s_{12}s_{34} + 8s_{15}s_{23}s_{12} + 8s_{15}s_{12}s_{45} + s_{12}s_{34}s_{45},
\end{aligned}$$

and

$$\begin{aligned}
B &= \frac{s_{12}}{s_{15}}\,(s_{34}+s_{23})\left\{-\frac{s_{45}}{s_{23}} - \frac{s_{12}}{s_{34}} + 1\right\} + \frac{s_{15}}{s_{12}}\,(s_{34}+s_{45})\left\{-\frac{s_{23}}{s_{45}} - \frac{s_{15}}{s_{34}} + 1\right\}\\
&+ \frac{1}{s_{34}}\left\{-2\,s_{15}s_{45} - s_{15}s_{23} - s_{12}s_{45} - 2\,s_{12}s_{23} + 2\,s_{15}s_{12} + {s_{12}}^2 + {s_{15}}^2\right\}\\
&+ \frac{s_{12}s_{13}}{s_{23}} + \frac{s_{15}s_{14}}{s_{45}} - 4\,s_{12} - 4\,s_{15} + s_{34}.
\end{aligned}$$

These results are consistent with Bern, Kosower and Dixson's results. (The difference between this result and BDK's result in the factor $(\sqrt{2})^5$ come from the different normalization of T^a.) We reaffirmed the BDK's results for the $m_{5;1}(-,-,+,+,+)$ case and $m_{5;1}(-,+,-,+,+)$ case, too.

6 Conclusion

A combination of the background field gauge, color decomposition and spinor helicity basis is presented as a very powerful method for the loop calculations in perturbative QCD. The method takes advantage of the three facts: 1) The simple structure of the color ordered Feynman rule in the background field gauge suppresses the number of the terms in the intermediate stages of the loop calculations. 2)Using the color decomposition technique, only a color-ordered subset of all possible Feynman diagrams is required. 3) Appropriate choices of the reference momenta in the helicity basis reduces the usually complicated tensor structure to a much simplified Dirac algebra. This also contributes in suppressing the number of the intermediate terms. These advantages simplify the loop calculation and makes multi-jet amplitude calculations at one-loop order feasible. This simplicity has been demonstrated in the example of five-gluon amplitude at the one loop level [7].

Since the method is formulated solely within the conventional gauge field theory, it is applicable to a wider range of calculations than one-loop perturbative QCD. For example, the extension of the background field gauge to theories with spontaneously broken symmetry like the electroweak theory is straightforward [23]. In this case, the Feynman rule still possesses the simple structure. We expect the method simplifies two-loop calculations, too.

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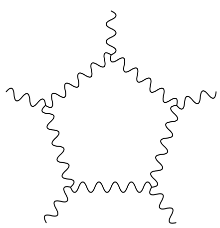
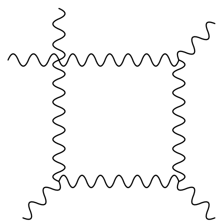
A Feynman diagrams for the one-loop five gluon vertex

Here we give the Feynman diagrams which contribute to the one-loop level five gluon amplitudes. We only show the gluonic loop diagrams. In the actual calculation, we need the ghost loop diagrams, too. We do not consider the renormalization of the wave functions. In the dimensional regularization, the diagrams 9) and 13) give no contribution.

1)



2)



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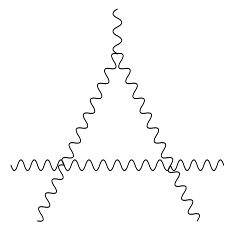

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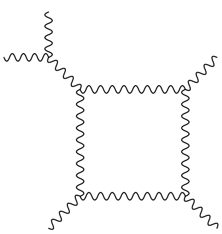

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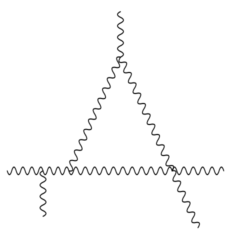

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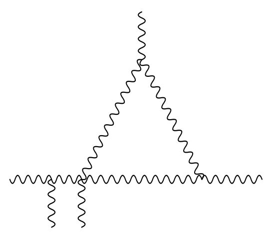

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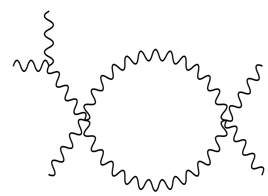
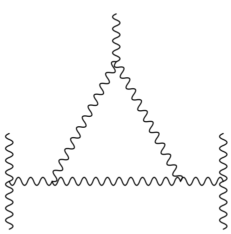
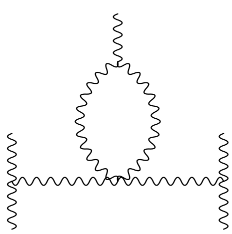


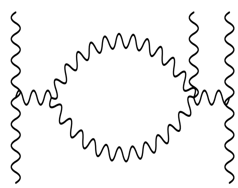
8)



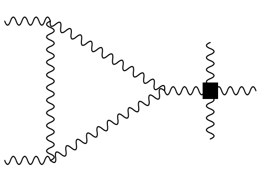
9)



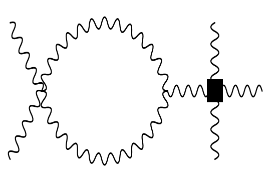
10)



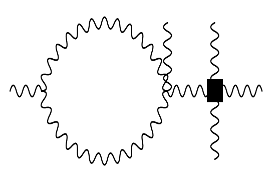
11)



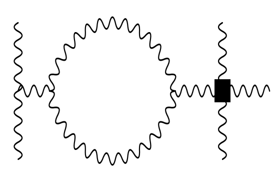
12)



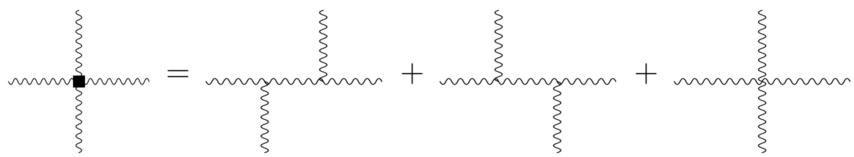
13)



14)



Here,



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